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The stability of plane Couette flow with viscous heating

By PETER C. SUKANEK
Department of Chemical Engineering, University of Massachusetts, Amherst

CHARLES A. GOLDSTEIN
Department of Chemical Engineering, Princeton University

AND ROBERT L. LAURENCE
Department of Chemical Engineering, University of Massachusetts, Amherst

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An investigation of the stability of plane Couette flow with viscous heating of a Navier–Stokes–Fourier fluid with an exponential dependence of viscosity upon temperature is presented. Using classical small perturbation theory, the stability of the flow can be described by a sixth-order set of coupled ordinary differential equations. Using Galerkin's method, these equations are reduced to an algebraic eigenvalue problem. An eigenvalue with a negative real part means that the flow is unstable.

Neutral stability curves are determined at Brinkman numbers of 15, 19, 25, 30, 40, 80 and 600 for Prandtl numbers of 1, 5 and 50. A Brinkman number of 19 corresponds approximately to the maximum shear stress which can be applied to the system.

The results indicate that four different modes of instability occur: one termed an inviscid mode, arising from an inflexion point in the primary flow; a viscous mode, due to the stratification of viscosity in the flow field and an associated diffusive mechanism; a coupling mode, resulting from the convective and viscous dissipation terms in the energy equation; and finally a purely thermal mode.

1. Introduction

Plane Couette flow, that is, simple shear flow between two infinite parallel flat plates, is the most elementary type of laminar motion, and probably the system most widely studied by investigators of hydrodynamic stability. All existing investigations tend to show that the flow is always stable to small disturbances for a Newtonian fluid with constant physical properties.

The behaviour of infinitesimal disturbances of plane Couette flow has been examined most recently by Gallagher & Mercer (1962, 1964), who employed a method of analysis similar to that used in the present work. They first calculated the lowest eigenvalue for the problem and later examined the higher eigenvalues. Their results, which are almost identical to those of Southwell & Chitty (1930), showed that the flow is always stable for finite Reynolds numbers. Deardorff (1963) arrived at the same conclusion. Ponomarenko (1968), using asymptotic and numerical methods, showed that the flow is stable as the Reynolds number approaches infinity. Finally, Gallagher (1969) examined a proof by Petrov (1940) which showed that both plane Couette and plane Poiseuille flow are always stable. Gallagher found Petrov's proof to be incorrect, but was able to recover the result for plane Couette flow.

Gill (1965) has shown that only a small change in the undisturbed velocity profile is required to change the profile from a stable one to an unstable one. The change must be such as to produce a local maximum in the magnitude of the vorticity.

Such a condition exists for the plane Couette flow of a Newtonian fluid with an exponential dependence of viscosity on temperature. Joseph (1964) and Gavis & Laurence (1968) have solved the equations defining the steady temperature and velocity profiles in such a case. They found that these profiles are not given uniquely by the shear stress. They are, however, given uniquely by the Brinkman number. These authors also found that there is a maximum shear stress which can be applied to the system.

Joseph (1965) investigated the stability of plane Couette flow with viscous heating. In the inviscid limit the velocity profile is stable along the low branch of the shear stress curve and unstable on the high branch. However, Joseph did not attempt to determine the stability characteristics at finite Reynolds numbers.

Goldstein (1968) analysed the low Reynolds number case and also found that instability can occur. This paper represents an extension of Goldstein's analysis and attempts to describe the neutral stability surface for the flow in the four-dimensional space of wavenumber, Reynolds, Prandtl and Brinkman numbers. In § 2, we state the equations for the steady temperature and velocity profiles, develop those for the infinitesimal disturbances and show how these equations can be solved. In § 3 we discuss the accuracy of the method and present the neutral stability curves for the flow. In § 4 we examine the limiting cases of flows with constant viscosity and no viscous heating. Finally, the physical bases of the instability and the shape of the neutral stability surfaces are discussed in § 5.

2. Formulation of the problem

In the analysis of the stability characteristics of plane Couette flow with viscous heating, we employ small perturbation theory. Assuming that the velocity, temperature and pressure fields suffer small deviations from the primary fields, we find the equations describing these deviations. The resulting differential eigenvalue problem can be reduced to an algebraic one using Galerkin's method.

In the mathematical formulation of this stability problem, we shall impose the following restrictions on the fluid.

(i) The fluid is incompressible.

(ii) The shear stress is a linear isotropic function of the rate of strain (Newtonian or Navier–Stokes fluid).

(iii) The heat flux is a linear isotropic function of the temperature gradient (Fourier fluid).

(iv) All physical properties except viscosity are constant.

(v) The viscosity is an exponential function of temperature.

Less restrictive assumptions could be taken. In particular, we could include buoyancy effects. Gallagher & Mercer (1965) and Ingersoll (1966) examined the plane Couette flow of a Newtonian liquid with constant viscosity and variable density subjected to a temperature gradient. Müller (1968) studied such a flow for a non-Newtonian fluid. All these investigations show that convective instability occurs at some critical value of the Rayleigh number. Below the critical value, the flow is stable for any Reynolds number.

It is not our purpose to examine here the convective instability problem. Rather, we wish to show that temperature-dependent viscosity and viscous dissipation alone can cause the flow to become unstable at a finite Reynolds number. For many Newtonian liquids of interest, the variation of thermal conductivity and density with temperature is small compared with variations of viscosity. Hence, assumptions (iv) and (v) do not represent stringent limitations on the applicability of the present analysis.

2.1. *The steady profiles*

The viscosity–temperature relation which we shall use was shown by Nahme (1940) to approximate closely the variations of viscosity of many liquids over a wide temperature range:

$$\mu = \mu_0 \exp\{-\beta(T-T_0)/T_0\}. \tag{2.1}$$

In this equation, β is an empirical constant and μ_0 is the viscosity at the reference temperature T_0. For liquids, β is a positive number, and we restrict ourselves to this case. In the plane Couette system we consider here, the upper plate moves in the $+x_1$ direction with velocity V_0. The plates are separated by a distance h, and both are at a constant temperature T_0.

Gavis & Laurence solved the equations for conservation of linear momentum and energy for this system. If we define the dimensionless variables

$$U = \frac{\bar{v}}{V_0}, \quad \theta = \frac{\beta(T-T_0)}{T_0}, \quad Br = \frac{\beta\mu_0 V_0^2}{kT_0}, \quad \eta = \frac{x_2}{h}, \tag{2.2}$$

where $\bar{v}$ is the fluid velocity in the $+x_1$ direction, their results become

$$\theta = \ln(a\,\mathrm{sech}^2 b\eta), \quad U = \tfrac{1}{2}(1+c\tanh b\eta), \tag{2.3}$$

where

$$a = 1+\tfrac{1}{8}Br, \quad b = \sinh^{-1}(\tfrac{1}{8}Br)^{\frac{1}{2}}, \quad c = \left(\frac{1+\frac{1}{8}Br}{\frac{1}{8}Br}\right)^{\frac{1}{2}}. \tag{2.4}$$

2.2. *The stability equations*

To analyse the stability of the flow subject to infinitesimal disturbances, we assume that the velocity, temperature and pressure consist of a steady part plus a small deviation from the steady value:

$$v_i = \bar{v}_i + \delta v_i', \quad T = \bar{T} + \delta T', \quad p = \bar{p} + \delta p'. \tag{2.5}$$

The overbar indicates that the quantity is time independent, and δ is a measure of the magnitude of the time-dependent disturbance, denoted by a prime.

Since the viscosity is an explicit function of the temperature and depends implicitly on position, it can be expanded in a Taylor series about the steady temperature $\bar{T}$. Using (2.1), we can write

$$\mu = \bar{\mu}(1-\beta\,\delta T'/T_0) + O(\delta^2). \qquad (2.6)$$

By substituting (2.1) and (2.2) into the conservation of mass, linear momentum and energy equations, subtracting the equations for the primary flow, and neglecting all terms nonlinear in δ, we arrive at the conservation equations for the small disturbances. Defining the dimensionless variables:

$$\left.\begin{array}{lll} u_i = v_i'/V_0, & \eta_i = x_i/h, & g = \bar{\mu}/\mu_0, \\ \vartheta = \beta T'/T_0, & \tau = V_0 t/h, & \mathscr{P} = p'/\rho V_0^2, \\ U_i = \bar{v}_i/V_0, & Re = \rho V_0 h/\mu_0, & Pe = \rho C_p Y_0 h/k, \end{array}\right\} \qquad (2.7)$$

these equations become†

$$\partial u_i/\partial \eta_i = 0, \qquad (2.8)$$

$$\frac{\partial u_i}{\partial \tau} + U_j\frac{\partial u_i}{\partial \eta_j} + u_j\frac{\partial U_i}{\partial \eta_j} = -\frac{\partial \mathscr{P}}{\partial \eta_i} + \frac{1}{Re}\left\{\frac{\partial}{\partial \eta_j}\left(g\left[\left(\frac{\partial u_i}{\partial \eta_j}+\frac{\partial u_j}{\partial \eta_i}\right) - \vartheta\left(\frac{\partial U_i}{\partial \eta_j}+\frac{\partial U_j}{\partial \eta_i}\right)\right]\right)\right\}, \qquad (2.9)$$

$$\frac{\partial \vartheta}{\partial \tau} + U_j\frac{\partial \vartheta}{\partial \eta_j} + u_j\frac{\partial \theta}{\partial \eta_j} = \frac{1}{Pe}\frac{\partial}{\partial \eta_j}\left(\frac{\partial \vartheta}{\partial \eta_j}\right)$$
$$+\frac{Br}{Pe}g\left\{\left[\frac{\partial u_i}{\partial \eta_j}+\frac{\partial u_j}{\partial \eta_i}\right]\frac{\partial U_i}{\partial \eta_j} + \left[\frac{\partial U_i}{\partial \eta_j}+\frac{\partial U_j}{\partial \eta_i}\right]\frac{\partial u_i}{\partial \eta_j} - \vartheta\left[\frac{\partial U_i}{\partial \eta_j}+\frac{\partial U_j}{\partial \eta_i}\right]\frac{\partial U_i}{\partial \eta_j}\right\}. \qquad (2.10)$$

For the isoviscous case, Squire (1933) has shown that the critical Reynolds number for two-dimensional disturbances is less than that for three-dimensional ones. We show in appendix A that this is not the case for the viscous heating problem. No simple transformation which could reduce the three-dimensional problem to a two-dimensional one exists.

The three-dimensional disturbances increase the complexity of the stability problem. Consideration of the simplest case, however, should precede a more complete analysis. As Lees & Lin (1946) first examined the stability to two-dimensional disturbances of compressible boundary layers, for which a Squire's theorem cannot be proved, and, later, Dunn & Lin (1953) examined the complete three-dimensional case, we shall ignore, for the present, the three-dimensional problem and consider only two-dimensional disturbances.

The next step in stability analysis is the introduction of the standard normal-mode expansion for the time-dependent disturbances. Assuming, then, that the disturbances u_i, ϑ and $\mathscr{P}$ are of the form

$$q = \hat{q}(\eta_2)\, e^{i\alpha(\eta_1 - \xi\tau)}, \qquad (2.11)$$

we obtain a set of four equations governing the dimensionless amplitude functions. These four equations are further reduced to a set of two coupled equations by the usual methods. Dropping the caret and letting $u_2 = v$, we arrive at

$$D^2(gD^2v) - 2\alpha^2 D(gDv) + \alpha^2(D^2g)\,v + \alpha^4 gv + i\alpha Re(D^2U)\,v$$
$$- i\alpha Re(U-\xi)(D^2-\alpha^2)v + i\alpha(gDU)(D^2+\alpha^2)\vartheta = 0 \qquad (2.12)$$

† Here and throughout this paper, summation over repeated indices is implied.

and
$$i\alpha(D^2-\alpha^2)\vartheta - Brg(DU)^2 i\alpha\vartheta + \alpha^2(U-\xi)\,Pe\vartheta = i\alpha v Pe D\theta + 2Br(gDU)(D^2+\alpha^2)v, \qquad (2.13)$$

where $D = d/d\eta_2$. These equations must be solved subject to the boundary conditions
$$v = Dv = \vartheta = 0 \quad \text{at} \quad \eta_2 = \pm 1. \qquad (2.14)$$

A change of variable simplifies the equations further. If we let
$$\left.\begin{aligned} \mathscr{T} &= i\alpha\vartheta, \quad U = \tfrac{1}{2}(1+\mathscr{U}), \\ \Omega &= gDU = \text{constant}, \quad -\tfrac{1}{2}\zeta = \tfrac{1}{2}-\xi, \quad \eta_2 = \eta, \end{aligned}\right\} \qquad (2.15)$$

equations (2.3) and (2.12)–(2.14) become
$$\mathscr{U} = c\tanh b\eta, \qquad (2.16)$$

$$D^2(gD^2v) - 2\alpha^2 D(gDv) + \alpha^2(D^2g)v + \alpha^4 gv + \tfrac{1}{2}i\alpha Re(D^2\mathscr{U})v - \tfrac{1}{2}i\alpha Re(\mathscr{U}-\zeta)(D^2-\alpha^2)v + \Omega(D^2+\alpha^2)\mathscr{T} = 0, \qquad (2.17)$$

$$(D^2-\alpha^2)\mathscr{T} - \tfrac{1}{2}\Omega Br(D\mathscr{U})\mathscr{T} - \tfrac{1}{2}i\alpha Pe(\mathscr{U}-\zeta)\mathscr{T} - i\alpha PeD\theta v - 2\Omega Br(D^2+\alpha^2)v = 0 \qquad (2.18)$$

and
$$v = Dv = \mathscr{T} = 0 \quad \text{at} \quad \eta = \pm 1. \qquad (2.19)$$

2.3. *Reduction to an algebraic eigenvalue problem*

Equations (2.17)–(2.19) define an eigenvalue problem. If the imaginary part of the eigenvalue ζ is positive, the flow will be unstable.

An approximate solution to the stability problem can be found using Galerkin's method (Ames 1965, p. 243). Finlayson (1968) and Finlayson & Scriven (1968) give examples of the Galerkin method applied to other stability problems. The variables v and $\mathscr{T}$ are expanded in complete sets of orthogonal functions which satisfy the boundary conditions. The coefficients of these functions are chosen by forcing the errors resulting from the substitution of these functions into the original differential equations to be orthogonal to the trial functions in the domain of interest.

The velocity disturbance v is expanded in a set of functions described by Reid & Harris (1958) and Chandrasekhar (1961, p. 643):
$$v_N = \sum_{n=1}^{N} \{a_n S_n(\eta) + b_n C_n(\eta)\}, \qquad (2.20)$$

where
$$\left.\begin{aligned} S_n(\eta) &= \frac{\sinh\mu_n\eta}{\sinh\mu_n} - \frac{\sin\mu_n\eta}{\sin\mu_n}, \\ C_n(\eta) &= \frac{\cosh\lambda_n\eta}{\cosh\lambda_n} - \frac{\cos\lambda_n\eta}{\cos\lambda_n}, \end{aligned}\right\} \qquad (2.21)$$

and μ_n and λ_n are given by the roots of the equations
$$\tanh\lambda + \tan\lambda = 0, \quad \coth\mu + \cot\mu = 0. \qquad (2.22)$$

Both $\{S_n\}$ and $\{C_n\}$ form orthogonal sets on the interval $[-1, +1]$. The set $\{S_n\}$ is everywhere orthogonal to the set $\{C_n\}$:

$$\left.\begin{aligned} \int_{-1}^{+1} C_m(n)\, C_n(\eta)\, d\eta = \int_{-1}^{+1} S_m(\eta)\, S_n(\eta)\, d\eta = 2\delta_{mn}, \\ \int_{-1}^{+1} C_m(\eta)\, S_n(\eta)\, d\eta = 0. \end{aligned}\right\} \tag{2.23}$$

The temperature disturbance $\mathscr{T}$ is expanded in a Fourier series:

$$\mathscr{T}_N = \sum_{n=1}^{N} \{c_n \sin p_n \eta + d_n \cos p_n \eta\}, \tag{2.24}$$

where $p_n = \frac{1}{2}(2n-1)\pi$.

If the series approximations to v and $\mathscr{T}$, equations (2.20) and (2.24), are substituted into (2.17) and (2.18), there will be a residue. If we call ϵ'_N the error in (2.17) and ϵ''_N that in (2.18), Galerkin's method requires that the constants a_n, b_n, c_n and d_n be such as to satisfy the equations

$$\left.\begin{aligned} \int_{-1}^{+1} \epsilon'_N S_j(\eta)\, d\eta = 0, \quad \int_{-1}^{+1} \epsilon'_N C_j(\eta)\, d\eta = 0, \\ \int_{-1}^{+1} \epsilon''_N \sin p_j \eta\, d\eta = 0, \quad \int_{-1}^{+1} \epsilon''_N \cos p_j \eta\, d\eta = 0. \end{aligned}\right\} \tag{2.25}$$

By performing the integrations in equations (2.25) and using the definitions in appendix B, the differential stability problem is reduced to an algebraic problem. Equations (2.25) can then be written in matrix form as

$$\begin{pmatrix} \mathbf{A}_1 - \lambda \mathbf{A}_2 & i\mathbf{B}_1 & \mathbf{C}_1 & 0 \\ -i\mathbf{A}_3 & \mathbf{B}_2 - \lambda \mathbf{B}_3 & 0 & \mathbf{D}_1 \\ \mathbf{C}_1^T & i\mathbf{B}_4 & \mathbf{C}_2 - \lambda \mathbf{C}_3 & i\mathbf{D}_2 \\ -i\mathbf{A}_5 & \mathbf{D}_1^T & i\mathbf{D}_2^T & \mathbf{D}_3 - \lambda \mathbf{C}_3 \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \mathbf{0}, \tag{2.26}$$

or

$$\mathbf{G}\mathbf{g} = \mathbf{0}, \tag{2.27}$$

where

$$\lambda = i\zeta. \tag{2.28}$$

The matrix $\mathbf{G}$ in (2.27) is asymmetric with complex elements. Gallagher & Mercer (1962) considered an analogous problem and suggested a transformation to map the matrix $\mathbf{G}$ into one having real elements. A similar transformation is employed here.

Consider the complex diagonal matrix

$$\Xi = \begin{pmatrix} \mathbf{I} & 0 & 0 & 0 \\ 0 & i\mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & i\mathbf{I} \end{pmatrix}, \tag{2.29}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix. We can define a new vector

$$\bar{\mathbf{g}} = \Xi^{-1}\mathbf{g}. \tag{2.30}$$

Also, let

$$\mathbf{H} = \begin{pmatrix} \mathbf{A}_2^{-1} & 0 & 0 & 0 \\ 0 & \mathbf{B}_2^{-1} & 0 & 0 \\ 0 & 0 & \mathbf{C}_3^{-1} & 0 \\ 0 & 0 & 0 & \mathbf{C}_3^{-1} \end{pmatrix}. \tag{2.31}$$

Premultiplying (2.27) by $\mathbf{H\Xi}$ and using (2.30), we have

$$\mathbf{H\Xi G\Xi}^{-1}\bar{\mathbf{g}} = 0, \tag{2.32}$$

or

$$\begin{pmatrix} \mathbf{A}_2^{-1}\mathbf{A}_1-\lambda\mathbf{I} & \mathbf{A}_2^{-1}\mathbf{B}_1 & \mathbf{A}_2^{-1}\mathbf{C}_1 & 0 \\ \mathbf{B}_3^{-1}\mathbf{A}_3 & \mathbf{B}_3^{-1}\mathbf{B}_2-\lambda\mathbf{I} & 0 & \mathbf{B}_3^{-1}\mathbf{D}_1 \\ \mathbf{C}_3^{-1}\mathbf{C}_1^T & \mathbf{C}_3^{-1}\mathbf{B}_4 & \mathbf{C}_3^{-1}\mathbf{C}_2-\lambda\mathbf{I} & \mathbf{C}_3^{-1}\mathbf{D}_2 \\ \mathbf{C}_3^{-1}\mathbf{A}_5 & \mathbf{C}_3^{-1}\mathbf{D}_1^T & -\mathbf{C}_3^{-1}\mathbf{D}_2^T & \mathbf{C}_3^{-1}\mathbf{D}_3-\lambda\mathbf{I} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{a}} \\ \bar{\mathbf{b}} \\ \bar{\mathbf{c}} \\ \bar{\mathbf{d}} \end{pmatrix} = 0. \tag{2.33}$$

The stability parameters λ are nothing more than the eigenvalues of the real matrix

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{A}_2^{-1}\mathbf{A}_1 & \mathbf{A}_2^{-1}\mathbf{B}_1 & \mathbf{A}_2^{-1}\mathbf{C}_1 & 0 \\ \mathbf{B}_3^{-1}\mathbf{A}_3 & \mathbf{B}_3^{-1}\mathbf{B}_2 & 0 & \mathbf{B}_3^{-1}\mathbf{D}_1 \\ \mathbf{C}_3^{-1}\mathbf{C}_1^T & \mathbf{C}_3^{-1}\mathbf{B}_4 & \mathbf{C}_3^{-1}\mathbf{C}_2 & \mathbf{C}_3^{-1}\mathbf{D}_2 \\ \mathbf{C}_3^{-1}\mathbf{A}_5 & \mathbf{C}_3^{-1}\mathbf{D}_1^T & -\mathbf{C}_3^{-1}\mathbf{D}_2^T & \mathbf{C}_3^{-1}\mathbf{D}_3 \end{pmatrix}. \tag{2.34}$$

From the definition of λ, equation (2.28), the stability of the flow depends upon the sign of the real part of λ. If it is positive, the flow will be stable. However, if it is negative, the flow will be unstable with respect to that particular mode, and the disturbance will grow until nonlinear effects dominate.

The eigenvalue problem was solved on the CDC 3600 digital computer. All the integrals were evaluated numerically using the Gaussian quadratures. The eigenvalues of $\mathbf{\Lambda}$ were found by the Q–R transformation, as described by Francis (1961). The details of the computer program can be found in Sukanek (1970).

3. Results

As opposed to the differential eigenvalue problem, which possesses an infinite set of eigenvalues, the algebraic problem yields only a finite number of characteristic values, the number depending on how many terms are used in the functions approximating the velocity and temperature disturbances. If the real part of any one of these eigenvalues is negative, the flow will be unstable. Therefore, in the search for neutrally stable modes, we have to be concerned only with the eigenvalue λ_0 with the least positive real part.

We found that the numerical procedure used to calculate the eigenvalues became unstable for N greater than six or eight. However, for N greater than four, as N increased, λ_0 varied only slightly. Also, the time required to find the eigenvalues increased as N^3. We felt that a good compromise between computing time and precision of results would be obtained by taking N as four. This means

αRe	α	λ_0 (G & M)	λ_0 (present)
0·125	0	158	157
	1	149	149
	2·5	187	187
1·25	0	15.8	15·8
	1·075	14·8	14·9
	3·225	24·1	24·1
12·5	0	1·55	1·59
	2·32	1·88	1·81
	4·64	4·27	4·13

TABLE 1. Comparison of present results with those of Gallagher & Mercer (1962)

that a total of sixteen eigenvalues were calculated for each case, requiring 2–2·5 s of computer time per case.

The real part of λ_0 approaches a limiting value from above as the number of terms increases. This indicates that the approximation to the true value of λ_0 is conservative. Therefore, the results presented here correspond to an upper bound to the solution of the stability problem.

As a check on the accuracy of the Galerkin method and the computer program for solving this stability problem, the program was modified to examine the stability of isoviscous plane Couette flow, a problem often attempted and well documented in the literature. Table 1 shows the results of these calculations. The lowest eigenvalues, all real for the set of parameters considered, for several values of wavenumber and αRe, the product of wavenumber and Reynolds number, are compared with the results of Gallagher & Mercer (1962, denoted by G & M). The percentage difference between our eigenvalues and theirs varies from 0 to less than 4 %.

Assured of the precision and accuracy of the methods, we proceed with the investigation of the stability characteristics of plane Couette flow with viscous heating. The points of neutral stability, that is, points where the real part of λ_0 is zero, were computed in a straightforward manner. The Prandtl and Brinkman numbers and αRe were fixed and a Fibonacci search (Wilde & Beightler 1967, pp. 236–241) was used to find the corresponding wavenumbers for neutral stability. Sixteen Fibonacci numbers were used, determining α to approximately three decimal places.

Curves of neutral stability computed in this manner were calculated at Brinkman numbers of 15, 19, 25, 30, 40, 80 and 600 for Prandtl numbers of 1, 5 and 50. The maximum shear stress which can be applied to the system corresponds to a Brinkman number of 18·214.†

For Brinkman numbers lower than 15, the numerical procedure becomes unstable because of the large values of αRe at the onset of instability. Hence, although the flow can be unstable at lower Brinkman numbers, neutral stability

† From the work of Gavis & Laurence, this Brinkman number is given by the solution of the equation, $\frac{1}{8}Br = \sinh^2[(8+Br)/Br]^{\frac{1}{2}}$.

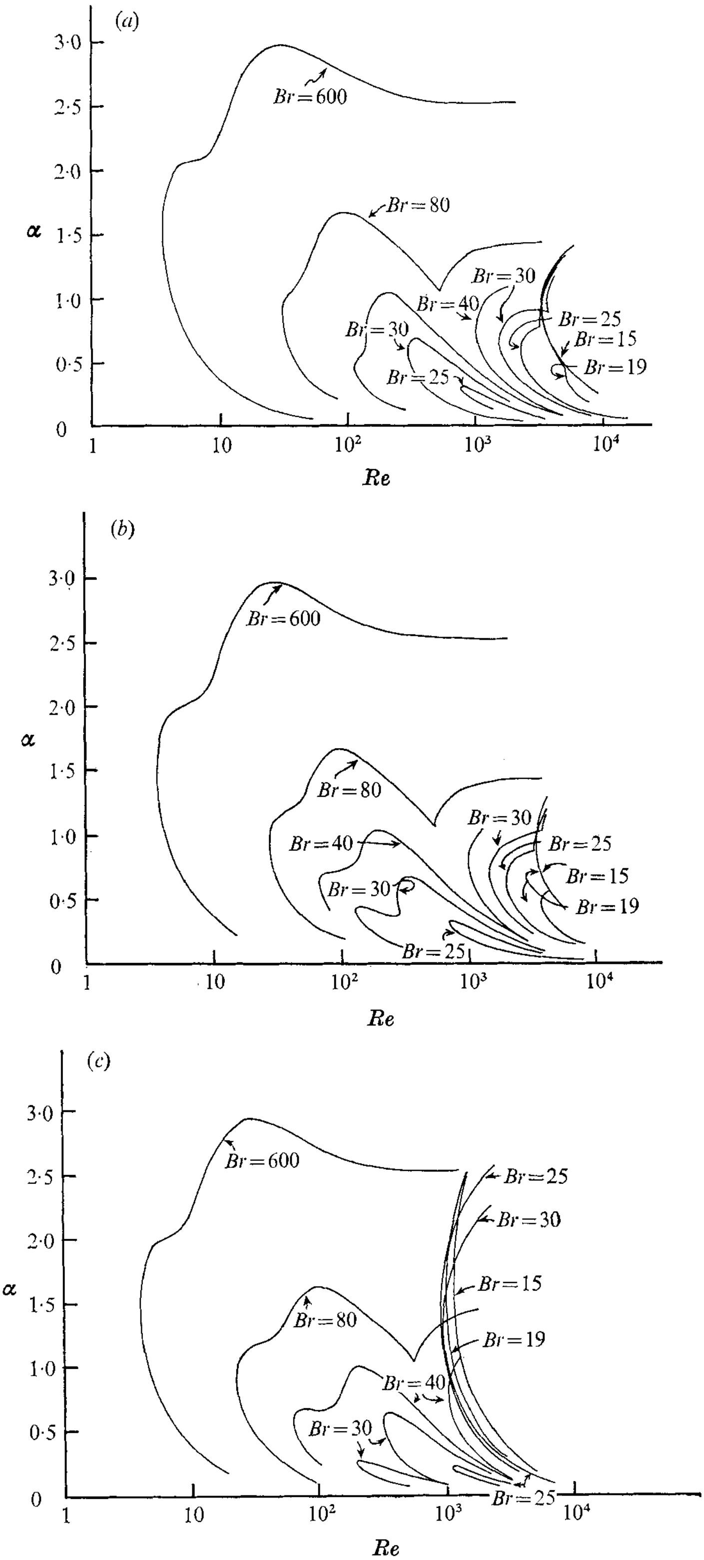

FIGURE 1. Neutral stability curves for (*a*) $Pr = 1{\cdot}0$, (*b*) $Pr = 5{\cdot}0$ and (*c*) $Pr = 50{\cdot}0$.

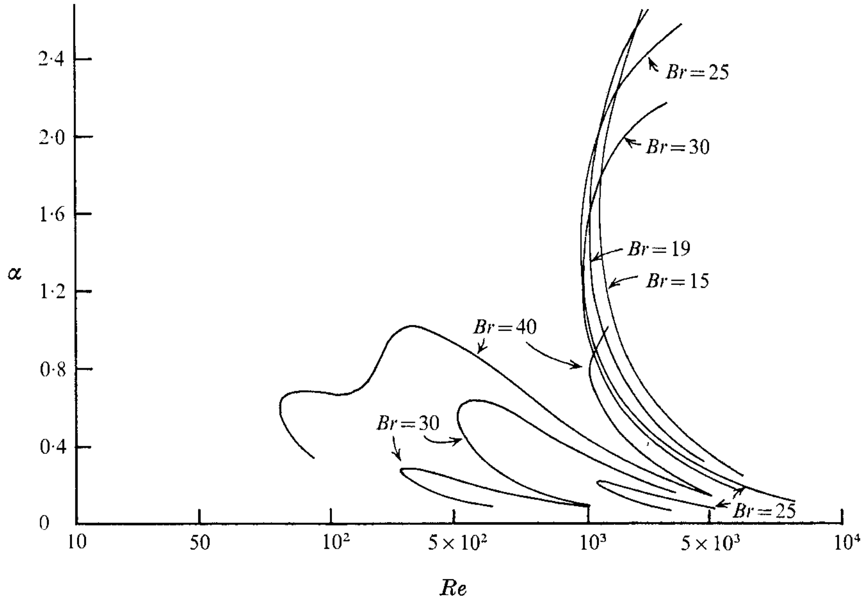

FIGURE 2. Neutral stability curves for $Pr = 50{\cdot}0$ for low and intermediate values of Br.

		Type 1		Type 2		Type 3		Type 4	
Br	Pr	Re_{crit}	α	Re_{crit}	α	Re_{crit}	α	Re_{crit}	α
15	1	—	—	—	—	—	—	3500	0·957
	5	—	—	—	—	—	—	3475	0·979
	50	—	—	—	—	—	—	1156	1.60
19	1	—	—	—	—	5240	0·410	3490	0·945
	5	—	—	—	—	3885	0·451	3455	0·955
	50	—	—	—	—	—	—	1028	1·535
25	1	—	—	791	0·291	2344	0·597	—	—
	5	—	—	726	0·317	2024	0·618	—	—
	50	—	—	1160	0·215	—	—	960	1·355
30	1	—	—	301	0·582	1695	0·655	3824	1·046
	5	136	0·406	289	0·519	1519	0·659	—	—
	50	191	0·283	322	0·544	—	—	980	1·225
40	1	116	0·473	—	—	1065	0·758	—	—
	5	68·2	0·645	—	—	1042	0·768	—	—
	50	64·2	0·623	—	—	(1043)	0·767)†	—	—
80	1	31·8	0·85	—	—	—	—	—	—
	5	27·7	0·975	—	—	—	—	—	—
	50	22·9	0·96	—	—	—	—	—	—
600	1	3·67	1·581	—	—	—	—	—	—
	5	3·92	1·527	—	—	—	—	—	—
	50	4·01	1·496	—	—	—	—	—	—

† This could be either type 3 or type 4 instability.

TABLE 2. Critical parameters

curves will not be presented because of the dubious nature of the numerical results for these cases.

The neutral stability curves are given in figures 1 (*a*), (*b*) and (*c*). Figure 2 is an enlarged version of the curves at low and intermediate Brinkman numbers for $Pr = 50$. Obviously, there are four different modes of instability. Correspondingly, there can be as many as four critical Reynolds numbers for given Brinkman and Prandtl numbers. However, for any one curve, no more than three could be determined. These critical values are given in table 2.

In all the calculations, except for high values of αRe, the eigenvalue with the least positive real part was purely real. For the higher Reynolds numbers, λ_0 occurred as a complex conjugate pair. In no case was the imaginary part equal to $\pm 0{\cdot}50$. Hence, from the definition of λ, equations (2.28) and (2.15), the first unstable mode is always oscillatory and the principle of exchange of stabilities does not apply.

4. Limits of constant viscosity and no viscous heating

After an examination of the general characteristics of the neutral stability curves for the viscous heating problem, one asks what is the nature of the stability of the isoviscous flow. Can the behaviour predicted by the Orr–Sommerfeld equation, which is a special case of the equations developed here, be recovered from this analysis?

To answer this question we shall consider two cases, constant viscosity ($\beta = 0$) and no viscous heating ($Br = 0$). Both limiting cases should be stable.

Taking first the limit of equations (2.16) and (2.17) as β approaches zero, we obtain the following set of equations governing the stability of isoviscous plane Couette flow with viscous heating:

$$D^4 v - 2\alpha^2 D^2 v + \alpha^4 v - \tfrac{1}{2} i\alpha Re(\eta - \zeta)(D^2 - \alpha^2) v = 0 \tag{4.1}$$

and $$(D^2 - \alpha^2)\mathscr{T}' - \tfrac{1}{2} i\alpha Pe(\eta - \zeta)\mathscr{T}' + \tfrac{1}{4} i\alpha Pe Br' \eta v - Br'(D^2 + \alpha^2) v = 0, \tag{4.2}$$

where $$\mathscr{T}' = \mathscr{T}/\beta, \quad Br' = Br/\beta. \tag{4.3}$$

Equation (4.1) is the Orr–Sommerfeld equation for plane Couette flow. It predicts stability at all Reynolds numbers. To determine the stability of the flow, we must examine the eigenvalues of the energy equation (4.2). This can be rewritten as

$$(D^2 - \alpha^2)\mathscr{T}' - \tfrac{1}{2} i\alpha Pe(\eta - \zeta)\mathscr{T}' = f(\eta). \tag{4.4}$$

To find the eigenvalues ζ, we need only look at the homogeneous equation. Multiplying this equation by the complex conjugate $\mathscr{T}'^*$ of $\mathscr{T}'$, and integrating from $\eta = -1$ to $\eta = +1$, we obtain

$$-K_2 + (\tfrac{1}{2} i\alpha Pe\zeta - \alpha^2) K_1 - \tfrac{1}{2} i\alpha Pe Q = 0, \tag{4.5}$$

where $$K_1 = \int_{-1}^{+1} |\mathscr{T}'|^2 d\eta, \quad K_2 = \int_{-1}^{+1} |D\mathscr{T}'|^2 d\eta, \quad Q = \int_{-1}^{+1} \eta \mathscr{T}'^* d\eta. \tag{4.6}$$

By adding to (4.5) its own complex conjugate and using the inequality

$$\tfrac{1}{2}i\alpha Pe(Q-Q^*) \leqslant \alpha Pe(K_1)^{\frac{1}{2}}, \tag{4.7}$$

we can bound Im (ζ) from above:

$$\mathrm{Im}\,(\zeta) \leqslant -\frac{2(K_2+\alpha^2 K_1)+\alpha Pe(K_1)^{\frac{1}{2}}}{\alpha Pe K_1}.$$

Since Im (ζ) is always negative, the flow is always stable.

In the limit of no viscous heating, the Brinkman number is zero. Stability is governed by the Orr–Sommerfeld equation, and again, the flow is stable.

It should be noted that the case of zero β is vastly different from the case of very small β. When the former is true, the primary velocity profile is a straight line, and the flow is always stable. For the latter case, however, the primary velocity profile possesses a point of inflexion, the viscosity becomes stratified throughout the flow field, the stability equations become coupled and the flow can be unstable.

5. Discussion

5.1. *Physical basis for instability*

The neutral stability curves shown in figure 1, especially for large Brinkman numbers, bear a striking resemblance to those given by Knowles & Gebhart (1968) and Nachtsheim (1963) for natural convection. The stability of their system is also described by a sixth-order set of coupled ordinary differential equations, the coupling being the result of temperature-dependent density and an imposed temperature difference. Gebhart (1971, pp. 377 ff.) attributes the bulge in the neutral stability curve to this coupling. Betchov & Criminale (1967, p. 193), in discussing Nachtsheim's results, suggest two modes of instability: one mode similar to the ordinary instability of an isothermal parallel flow and another coupling mode, depending on temperature fluctuations.

The neutral stability curves presented here show four different modes of instability. The various curves sketched in figure 3 can be thought of as being archetypal of these four different modes. Since none of the neutral stability curves calculated here show the four types of instability distinctly, we shall refer to this figure in discussing the different mechanisms of instability. There is one mode, which we shall call the inviscid mode, which is a result of having a primary velocity distribution with an inflexion point. The second mode, the viscous mode, results from the interaction of the variable viscosity with the viscous terms in the momentum equation.

A third mode exists because of the coupling of the two stability equations, through the convective terms and the viscous dissipation term in the energy equation. Finally, there is a fourth mode which is a purely thermal mode, and arises from the temperature equation.

Hereafter, we shall refer to these four modes simply as inviscid, viscous, coupling and thermal. We now discuss each in turn.

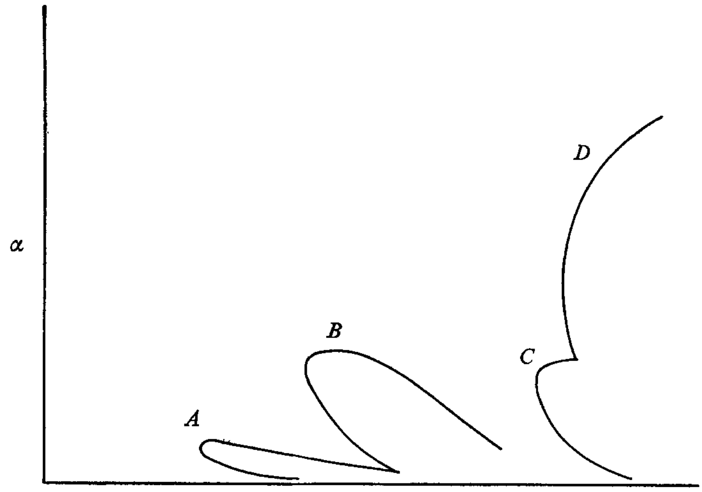

FIGURE 3. Idealized neutral stability curves.

The existence of the inviscid mode is a consequence of having a primary-flow velocity distribution of the form

$$U = b \tanh c\eta. \tag{5.1}$$

This type of profile is usually associated with shear layers (see, for example, Betchov & Criminale (1967, pp. 26–37, 74–83) and Esch (1957)). Profiles of this type, which possess an inflexion point at the centre-line, do not satisfy Rayleigh's theorem and do satisfy Fjortoft's theorem (Yih 1969, pp. 470–472), indicating that the flow may possibly be unstable in the inviscid limit.

Joseph (1965) analysed the viscous heating profile in the limit of infinite Reynolds number. As mentioned previously, he concluded that the velocity profile is stable along the lower branch of the shear stress curve, but unstable along the upper branch. Hence, for Brinkman numbers greater than 18·214, we may expect this inviscid type of instability to occur and to appear at finite values of the Reynolds number.

The second mode of instability, the viscous mode, has as its origin the variation of viscosity throughout the flow channel. As shown by the last term in (2.14), a 'diffusive' mechanism of viscosity variation is also present. Not only does the viscosity change continuously throughout the flow channel, but any mechanism which alters the position of a fluid particle in the flow field will also alter its viscosity.

Craik & Smith (1968) treated the problem of free-surface flows with viscosity stratification. Later, Craik (1969) analysed plane Couette flow with variable viscosity and allowed for the presence of an arbitrary diffusive mechanism. The latter investigation, which is of greater importance in this discussion, revealed a stabilizing or destabilizing mechanism which depends upon a viscosity gradient. This is in contrast to Yih's (1967) investigation of the stability of two superposed fluids of different viscosity, where instability results from a viscosity discontinuity. Craik found that the flow may be stable or unstable at very small Reynolds numbers to disturbances of long wavelength. The flow is stable if the second derivative of the velocity distribution resulting from the viscosity stratification is positive at the critical point, and unstable if the second derivative is negative. The critical point is, as usual, the point where the primary-flow velocity equals the real part of the complex wave speed.

In the present flow problem, the critical point occurs at the centre of the channel, where the second derivative is zero. From Craik's results, we cannot, therefore, conclude either stability or instability; only the existence of a neutral mode is assured. However, since this result is valid at very low Reynolds numbers, we might expect the viscosity stratification in the present case to be a cause of instability at higher, though finite, Reynolds numbers.

The first two modes result primarily from the momentum equation. We now turn to the coupling mode and the thermal mode, for which the energy equation is more important.

The presence of the coupling mode, as pointed out above, is a direct consequence of the interaction of the two equations. Similar types of instability have been noted in the analysis of natural convective flows, and are expected to occur in the present investigation since the governing equations are of the same nature.

That a fourth unstable mode exists is beyond question because of the character of the neutral stability curves. The explanation for cause of this mode, however, must remain purely speculative in nature. We shall attribute this mode to purely thermal effects. Joseph (1964) examined the stability of the temperature distribution. He concluded that the temperature profile is stable for Brinkman numbers less than 18·214, but unstable for Brinkman numbers above this critical value. Joseph, however, was not investigating the stability of the flow, since he treated only a one-dimensional disturbance to the diffusion equation. Hence, we may only conclude that a possible, and reasonable, cause is purely thermal in nature.

5.2. *Shape of the neutral stability surface*

The neutral stability surface governing plane Couette flow with frictional heating and temperature-dependent viscosity is best described in the four-dimensional space of wavenumber and Reynolds, Prandtl and Brinkman numbers.

To assign one of the above-mentioned modes of instability to the archetypes in figure 3, it is necessary to compare the various curves of figure 1 to ascertain to what extent the Prandtl number influences each Brinkman number curve.

Curve A shows a strong dependence upon Prandtl number for intermediate Brinkman numbers (25, 30 and 40), and varies but little at the higher Brinkman numbers. Curve B, for a given Brinkman number, is almost independent of the Prandtl number. Curve C behaves similarly to A: high Prandtl number dependence at lower Brinkman numbers, with the effect of Prandtl number decreasing as the Brinkman number is increased. Finally, curve D shows a most peculiar Prandtl number dependence. At low Prandtl numbers, the curves are essentially independent of Prandtl number. However, for the highest Prandtl number curves, the Prandtl number effect is quite significant.

An analysis of the calculated eigenvalues revealed that the two modes which occur at the lowest Reynolds number, curves A and B of figure 3, are characterized by having an eigenvalue with zero real part, i.e. ζ_0 is purely real. This means, from (2.28) and (2.15), that ξ_r is 0·5 for these points on the neutral stability curve. Since this is the velocity on the centre-line, it indicates that the disturbances which occur at the lowest Reynolds numbers arise from the critical layer in the flow field.

The two higher Reynolds number curves, curves C and D, on the other hand, are characterized by ζ_0 occurring in complex-conjugate pairs. The real part ξ_r of the complex wave speed is no longer equal to the centre-line primary-flow velocity, and the origin of instability moves out of the critical layer.

Consider first the inviscid mode. If the flow becomes unstable because of the velocity profile, we would expect this mode to originate at the critical layer, to be essentially independent of the Prandtl number and to occur only at Brinkman numbers above the critical value. The curve that fulfils these requirements is curve B. Furthermore, this curve moves to lower values in the Reynolds number plane as the Brinkman number is increased, which is what would be expected of this mode since the velocity profile is becoming more and more skewed.

The nature of the eigenvalues at neutral stability for curve A is the same as for curve B, but the parameter dependence is very different. There is a strong effect of Prandtl number at the lower Brinkman numbers, the flow becoming more unstable as Pr is increased. As the Brinkman number is further increased the effect of Prandtl number is diminished, and actually reverses at the highest Br. This behaviour may be explained by identifying this curve with the coupling mode. At low Brinkman numbers, the coupling results from the penultimate term in (2.15), the convective term. As the Prandtl number is increased, this term becomes more and more important, and the neutral curve moves to lower values of the Reynolds number. The Prandtl number is so important, in fact, that the coupling mode, absent for $Pr = 1$ and $Br = 30$, becomes the dominant mode as the Prandtl number is increased to 5. As the Brinkman number is increased, however, the last term in the energy equation, the viscous heating term, begins to dominate, and the effect of Prandtl number is lessened. Physically, at low Brinkman numbers, the Péclet number is the important parameter. Increasing the Prandtl number essentially means decreasing the thermal conductivity, and the heat conduction process in the fluid is decreased. At very high Brinkman numbers, an increase in Prandtl number corresponds to increasing the heat capacity of the fluid, thereby decreasing somewhat the magnitude of the thermal variations.

Curve C of figure 3 is most easily explained on the basis of a purely thermal mechanism. As the Prandtl number increases, the critical Reynolds number decreases, for any Brinkman number. However, the larger the Brinkman number, the less effect the Prandtl number has. This is the behaviour we would expect, if the first three terms in (2.15) are identified with this thermal mechanism. As the Brinkman number increases, the significance of this mode becomes less compared with that of the two previously mentioned modes.

Finally, we come to curve D, where shape is most easily explained by identifying this curve with the viscous mode. The neutral stability curve is almost independent of both the Brinkman number and Prandtl number at the lowest Prandtl number, but changes drastically as the Prandtl number is increased from 5 to 50, although the Brinkman number effect is still not very pronounced even at this high Prandtl number. Consider the cause of the viscous mode: stratification of viscosity, coupled with a diffusive mechanism, the energy equation. At low Prandtl numbers, the diffusive mechanism is not important, and this should be reflected in the neutral stability curves. As the Prandtl number is increased, however, the importance of the coupled energy equation is enhanced, and character of the neutral curve should change. Thus, the viscous mode is comprised of two different subclasses: one with and one without an associated diffusive mechanism. The former subclass should only be important at high Prandtl numbers. The fact that the curves are almost independent of Brinkman number can be explained by remembering that this mode is primarily due to the gradient of viscosity, which is not greatly changed as the Brinkman number varies by a factor of two from 15 to 30. As the Brinkman number increases further, however, this mode becomes relatively unimportant.

6. Conclusion

We have presented the neutral stability curves for a plane Couette flow of a Newtonian liquid with an exponential dependence of viscosity on temperature. Four different modes of instability, typified by the four neutral curves of figure 3, were found. The four curves A, B, C and D are associated with a coupling mode, an inviscid mode, a purely thermal mode and a viscous mode, respectively.

The primary flows described herein would not be difficult to generate experimentally. The Brinkman numbers, though they appear to be large at first glance, can easily be realized. It is a relatively simple task to choose a fluid with a large enough β and large enough μ_0 so that Brinkman and Reynolds numbers of the same order of magnitude as those given here can be attained. Our recent work on chlorinated polyphenyls has shown values of β from 30 to 90 and viscosities in the range of 1 to 10^5 poise. Such fluids also behave in a Newtonian manner up to very large shear rates.

The authors would like to express their thanks to the University Computing Centre, University of Massachusetts, for supplying the computer time required to perform all the numerical calculations.

Appendix A. Squire's theorem

Squire (1933) was able to show that the stability problem with three-dimensional disturbances is equivalent to a two-dimensional problem at a lower Reynolds number for isoviscous plane Couette flow. Obviously, it would be very helpful if a theorem analogous to Squire's held in the present analysis as well. Unfortunately, this is not the case. When only the momentum equation is considered, Squire's theorem is valid. However, because of the viscous dissipation terms, the theorem cannot be proved with regard to the energy equation.

The momentum equation

If, instead of equation (2.11), we assume three-dimensional disturbances for u_i and $\mathscr{P}$ of the form

$$q = \hat{q}(\eta_2)\exp\{i(\alpha\eta_1+\gamma\eta_3-\alpha\xi\tau)\} \tag{A 1}$$

and substitute (A 1) into (2.8) and (2.7), we arrive at an equation similar to (2.10). Again, letting $\hat{u}_2 = v$ we have

$$\begin{aligned}-i\alpha(U-\xi)\,[D^2-(\alpha^2+\gamma^2)]\,v+i\alpha vD^2U \\ +2Re^{-1}DgD[D^2-(\alpha^2+\gamma^2)]\,v+Re^{-1}(gDU)\,i\alpha[D^2+(\alpha^2+\gamma^2)]\,\vartheta \\ +Re^{-1}D^2g[D^2+(\alpha^2+\gamma^2)]\,v+gRe^{-1}[D^2-(\alpha^2+\gamma^2)]\,v = 0.\end{aligned} \tag{A 2}$$

Multiplying by $(\alpha^2+\gamma^2)^{\frac{1}{2}}/\alpha$, defining

$$\overline{\alpha}^2 = \alpha^2+\gamma^2, \quad \overline{Re} = \overline{\alpha}^{-1}Re, \quad \mathscr{T} = i\alpha\vartheta \tag{A 3}$$

and rearranging, we find that

$$\begin{aligned}D^2(gD^2v)-2\overline{\alpha}^2D(gDv)+\overline{\alpha}^2(D^2g)\,v+\overline{\alpha}^4gv+i\overline{\alpha}\overline{Re}(D^2U)\,v \\ -i\overline{\alpha}\overline{Re}(U-\xi)\,(D^2-\overline{\alpha}^2)\,v+gDU(D^2+\overline{\alpha}^2)\,\mathscr{T} = 0.\end{aligned} \tag{A 4}$$

This is exactly the equation satisfied by a two-dimensional disturbance. Since $\overline{Re}$ is necessarily less than Re, Squire's theorem is proved, at least for the momentum equation.

The energy equation

If we take ϑ as being a three-dimensional disturbance of the form (A 1), substitution of this into (2.10) gives

$$i\alpha(U-\xi)\,\vartheta+vD\theta = \frac{1}{Pe}[D^2-(\alpha^2+\gamma^2)]\,\vartheta+\frac{Br}{Pe}gDU[2Du+i\alpha v-\vartheta DU], \tag{A 5}$$

where u is the velocity in this η_1 direction.

There is no way to eliminate the Du term in this equation without introducing another extraneous term. The energy equation with three-dimensional disturbances cannot be reduced to the same form as the equation with two-dimensional disturbances.

We have arrived at a rather peculiar result. Squire's theorem is valid for the momentum equation, but not for the energy equation. Considering only two-dimensional disturbances is sufficient if the disturbances come from the velocity equation, but if the disturbances come from the energy equation, we must look at three-dimensional disturbances to obtain the critical Reynolds number.

Appendix B. Definitions of integrals

Substitution of the series approximation for v and $\mathscr{T}$, equations (2.20) and (2.24) into the conservation equations for the disturbances, equations (2.17) and (2.18), gives the errors ϵ'_N and ϵ''_N:

$$\sum_{n=1}^{N}\{a_n[D^2(gD^2S_n)-2\alpha^2D(gDS_n)+\alpha^2(D^2g)S_n$$
$$+\alpha^4gS_n+\tfrac{1}{2}i\alpha Re(D^2\mathscr{U})S_n-\tfrac{1}{2}i\alpha Re(\mathscr{U}-\zeta)(D^2-\alpha^2)S_n]$$
$$+b_n[D^2(gD^2C_n)-2\alpha^2D(gDC_n)+\alpha^2(D^2g)C_n$$
$$+\alpha^4gC_n+\tfrac{1}{2}i\alpha Re(D^2\mathscr{U})C_n-\tfrac{1}{2}i\alpha Re(\mathscr{U}-\zeta)(D^2-\alpha^2)C_n]$$
$$+c_n[\Omega(D^2+\alpha^2)\sin p_n\eta]+d_n[\Omega(D^2+\alpha^2)\cos p_n\eta]\}=\epsilon'_N, \tag{B 1}$$

$$\sum_{n=1}^{N}\{c_n[(D^2-\alpha^2)\sin p_n\eta-\tfrac{1}{2}\Omega Br(D\mathscr{U})\sin p_n\eta-\tfrac{1}{2}i\alpha Pe(\mathscr{U}-\zeta)\sin p_n\eta]$$
$$+d_n[(D^2-\alpha^2)\cos p_n\eta-\tfrac{1}{2}\Omega Br(D\mathscr{U})\cos p_n\eta-\tfrac{1}{2}i\alpha Pe(\mathscr{U}-\zeta)\cos p_n\eta]$$
$$+a_n[-i\alpha Pe(D\theta)S_n-2\Omega Br(D^2+\alpha^2)S_n]$$
$$+b_n[-i\alpha Pe(D\theta)C_n-2\Omega Br(D^2+\alpha^2)C_n]\}=\epsilon''_N. \tag{B 2}$$

Equations (B 1) and (B 2) are substituted into equations (2.25), and the integrations are performed. We define the integrals

$$\left.\begin{aligned}
Q_1^{nj}&=\int_{-1}^{+1}\cosh^2 b\eta(D^2S_n)(D^2S_j)\,d\eta, & Q_2^{nj}&=\int_{-1}^{+1}\cosh^2 b\eta\,(DS_n)(DS_j)\,d\eta,\\
Q_3^{nj}&=\int_{-1}^{+1}\cosh^2 b\eta S_nS_j\,d\eta, & Q_4^{nj}&=\int_{-1}^{+1}(DS_n)(DS_j)\,d\eta,\\
P_1^{nj}&=\int_{-1}^{+1}\operatorname{sech}^2 b\eta\tanh b\eta C_nS_j\,d\eta, & P_2^{nj}&=\int_{-1}^{+1}\tanh b\eta(D^2C_n)S_j\,d\eta,\\
P_3^{nj}&=\int_{-1}^{+1}\tanh b\eta C_nS_j\,d\eta, & P_4^{nj}&=\int_{-1}^{+1}\tanh b\eta C_jD^2S_n\,d\eta,\\
L_1^{nj}&=p_n\int_{-1}^{+1}(DS_j)\cos p_n\eta\,d\eta, & L_2^{nj}&=\int_{-1}^{+1}S_j\sin p_n\eta\,d\eta,\\
M_1^{nj}&=\int_{-1}^{+1}\cosh^2 b\eta(D^2C_n)(D^2C_j)\,d\eta, & M_2^{nj}&=\int_{-1}^{+1}\cosh^2 b\eta(DC_n)(DC_j)\,d\eta,\\
M_3^{n}&=\int_{-1}^{+1}\cosh^2 b\eta C_nC_j\,d\eta, & M_4^{nj}&=\int_{-1}^{+1}(DC_n)(DC_j)\,d\eta,\\
N_1^{nj}&=\int_{-1}^{+1}(-p_n)DC_j\sin p_n\eta\,d\eta, & N_2^{nj}&=\int_{-1}^{+1}C_j\cos p_n\eta\,d\eta,\\
O_2^{nj}&=\int_{-1}^{+1}\operatorname{sech}^2 b\eta\sin p_n\eta\sin p_j\,d\eta, & O_1^{nj}&=p_np_j\delta_{nj},\\
R_1^{nj}&=\int_{-1}^{+1}\tanh b\eta\cos p_n\eta\sin p_j\eta\,d\eta, & S_1^{nj}&=\int_{-1}^{+1}\tanh b\eta C_n\sin p_j\eta\,d\eta,\\
T_1^{nj}&=p_np_j\delta_{nj}, & T_2^{nj}&=\int_{-1}^{+1}\operatorname{sech}^2 b\eta\cos p_n\eta\cos p_j\eta\,d\eta,\\
U_1^{nj}&=\int_{-1}^{+1}\tanh b\eta\,S_n\cos p_j\eta\,d\eta.
\end{aligned}\right\} \tag{B 3}$$

These integrals are used in equations (2.25). A simplification of these equations results if we also define

$$
\left.\begin{aligned}
A_1^{nj} &= 2a^{-1}Br\{Q_1^{nj}+2\alpha^2Q_2^{nj}+\alpha^4Q_3^{nj}+4b^2\alpha^2(Q_3^{nj}-\delta_{nj})\},\\
A_2^{nj} &= \alpha ReBr\{Q_4^{nj}+2\alpha^2\delta_{nj}\},\\
B_1^{nj} &= 2cBr\alpha Re\{\tfrac{1}{2}\alpha^2P_3^{nj}-b^2P_1^{nj}-\tfrac{1}{2}P_2^{nj}\},\\
B_2^{nj} &= 2a^{-1}Br\{M_1^{nj}+2\alpha^2M_2^{nj}+\alpha^4M_3^{nj}+4b^2\alpha^2(M_3^{nj}-\delta_{nj})\},\\
B_3^{nj} &= \alpha ReBr\{M_4^{nj}+2\alpha^2\delta_{nj}\},\\
B_4^{nj} &= -2\alpha bPeS_1^{nj},\\
A_3^{nj} &= 2\alpha RecBr\{-\tfrac{1}{2}\alpha^2P_3^{jn}+b^2P_1^{jn}+\tfrac{1}{2}P_4^{nj}\},\\
A_5^{nj} &= 2b\alpha PeU_1^{nj},\\
C_1^{nj} &= cba^{-1}Br\{\alpha^2L_2^{nj}-L_1^{nj}\},\\
C_2^{nj} &= O_1^{nj}+\alpha^2\delta_{nj}+(b^2c^2/4a)\,BrO_2^{nj},\\
C_3^{nj} &= \tfrac{1}{2}\alpha Pe\delta_{nj},\\
D_1^{nj} &= cba^{-1}Br\{-N_1^{nj}+\alpha^2N_2^{nj}\},\\
D_2^{nj} &= \tfrac{1}{2}\alpha PecR_1^{nj},\\
D_3^{nj} &= T_1^{nj}+\alpha^2\delta_{nj}+(b^2c^2/4a)\,BrT_2^{nj}.
\end{aligned}\right\} \qquad (\mathrm{B}\,4)
$$

With equations (B 4), the integrals (2.25) become

$$
\begin{aligned}
a_n\{A_1^{nj}-\lambda A_2^{nj}\}+ib_nB_1^{nj}+c_nC_1^{nj} &= 0,\\
-ia_nA_3^{nj}+b_n\{B_2^{nj}-\lambda B_3^{nj}\}+d_nD_1^{nj} &= 0,\\
a_nC_1^{jn}+ib_nB_4^{nj}+c_n\{C_2^{nj}-\lambda C_3^{nj}\}+id_nD_2^{nj} &= 0
\end{aligned}
$$

and

$$
-i\alpha_nA_5^{nj}+b_nD_1^{jn}-ic_nD_2^{jn}+d_n\{D_3^{nj}-\lambda C_3^{nj}\} = 0. \qquad (\mathrm{B}\,5)
$$

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